J. Fluid Mech. (2002), *vol.* 464, *pp.* 209–215.
DOI: 10.1017/S0022112002001003 Printed in the United Kingdom

Viscous flow between two moving parallel disks: exact solutions and stability analysis

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(Received 12 February 2002)

The motion of a viscous incompressible liquid between two parallel disks, moving towards each other or in opposite directions, is considered. The description of possible conditions of motion is based on the exact solution of the Navier–Stokes equations. Both stationary and transient cases have been considered. The stability of the motion is analysed for different initial perturbations. Different types of stability were found according to whether the disks moved towards or away from each other.

1. Introduction

There is a large class of processes which can be considered from the mathematical point of view as the motion of liquid between two parallel disks, moving towards each other or in opposite directions with a constant velocity. These include such processes as the motion of liquid through a hydraulic pump, and the motion of underground water can also be described with a help of the current model. In figure 1 these two applications are presented. It should be noted that in spite of the different types of hydrodynamical problem at first sight, the mathematical descriptions are the same. So it is possible to describe the water motion in a hydraulic pump (when impermeable disks are moving toward or apart each other) similarly to the motion of underground water (when permeable disks are fixed). The second case refers to water motion through porous media. These problems are interesting because some of their solutions, though analytically obtained, can be confirmed by experiments.

For example, place two parallel disks in water and start moving them towards each other or in opposite directions, assuming the size of the disks to be much larger than the distance between them. Even with a qualitative assessment we can see that when the disks are approaching each other the effort required is smaller than that for separation when the disks are moving apart. This can be explained by the different character of the liquid motion: when the disks are approaching it is potential; when the disks are moving apart it is rotational.

This work deals with a description of the types of possible instability of such motion. Craik & Criminale (1986) described a procedure for finding classes of exact solutions of the Navier–Stokes equations. These solutions consist of a 'basic flow' with spatially uniform rates of strain and a 'disturbance' of a planar form: the disturbance is continuously distorted by the basic flow but nevertheless remains of planar form at all times. A somewhat similar formulation was given by Lagnado, Phan-Thien &

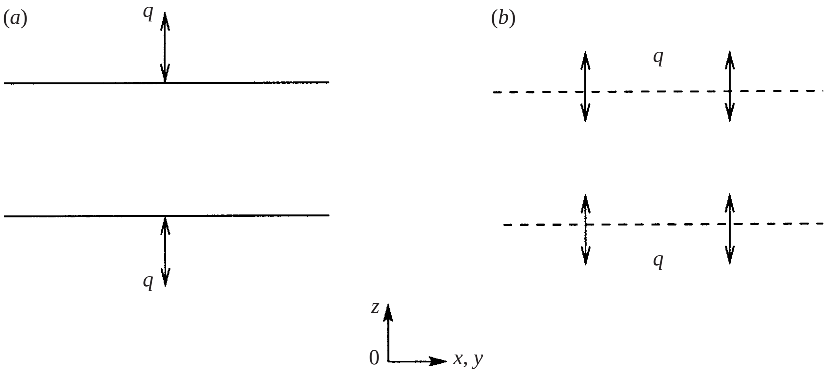

FIGURE 1. Different applications of the model: (*a*) moving impermeable disks and (*b*) fixed permeable disks.

Leal (1984), but was restricted to two-dimensional basic flows and the authors were unaware that their linearized approximation is in fact an exact solution for single plane-wave modes.

The aims of this paper are twofold. The first is to generalize the results of Craik (1989) in a case of plane-wave superposition. The second is to find the possible forms of the jet solutions which are generated as a result of the instability development.

The paper is organized as follows. In §2 the mathematical formulation of the problem is described. Section 3 contains a method of analysis the problem, while results and discussion are presented in §4. The paper ends with some concluding remarks.

2. Formulation of the problem

Consider the motion of viscous incompressible liquid induced by two parallel disks moving towards each other in the case when $h \ll l$ (where h is the distance between the disks, and l is the length of the disks). Let us assume that the horizontal velocity does not depend on the vertical coordinate whereas the vertical velocity depends linearly on the distance between the disks. In this case the Navier–Stokes equations have the following form (Craik 1989; Craik & Criminale 1986; Lagnado *et al.* 1984):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2q, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \nu \Delta u, \tag{2.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \nu \Delta v, \tag{2.3}$$

where the velocity components are represented as

$$v_x = u(x, y, t), \qquad v_y = v(x, y, t), \qquad v_z = -2qz, \qquad p = p(x, y, t) - 2q^2z^2. \tag{2.4}$$

Furthermore, p is the pressure divided by the liquid density, and q is the relative velocity of the disks, assumed here to be constant. It should be noted also that the equation for the vertical velocity coordinate v_z is identically equal to zero.

3. Method of analysis

For convenience of analysis let us select the potential component from the horizontal components of the velocity and introduce the flow function:

$$u = qx + \frac{\partial\psi}{\partial y}, \tag{3.1}$$

$$v = qy - \frac{\partial\psi}{\partial x}, \tag{3.2}$$

where ψ is the stream function. Now equation (2.1) is satisfied identically, and equations (2.2) and (2.3) together, after elimination of the pressure and introduction of the vorticity, will give the equations of motion in the following form:

$$\frac{\partial\omega}{\partial t} + \{\psi, \omega\} = -q\left(\frac{\partial}{\partial y}(y\omega) + \frac{\partial}{\partial x}(x\omega)\right) + \nu\Delta\omega, \tag{3.3}$$

where ω is the vorticity:

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \Delta\psi, \tag{3.4}$$

and $\{\psi, \omega\}$ denotes the Poisson brackets:

$$\{\psi, \omega\} = \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}. \tag{3.5}$$

One of the solutions of equation (3.3) is $\psi = 0$, which corresponds to liquid potential motion, known as the motion near the stagnation point (other solutions for ψ are given in §§ 4.2 and 4.3). Following the work of Craik (1989), to investigate the stability of this solution let us consider the periodical one-dimensional perturbation $\delta\psi$. This perturbation is expressed by the following equation:

$$\delta\psi = k^{-2}(t)A(t)\cos(k(t)x). \tag{3.6}$$

To analyse the change of the vorticity in the course of time we plug the stream function $\delta\psi$ (3.6) into (3.3) and equate groups of items with the same x-powers to obtain the following system of nonlinear equations:

$$\frac{\partial k}{\partial t} = -qk, \tag{3.7}$$

$$\frac{\partial A}{\partial t} = -qA - \nu Ak^2. \tag{3.8}$$

After solving the linear equation (3.7) the result is put into (3.8) and then the general solution is

$$A(t) = A(0)\exp\left(-qt + \frac{\nu k^2(0)}{2q}(-1 + \mathrm{e}^{-2qt})\right), \tag{3.9}$$

$$k(t) = k(0)\exp(-qt), \tag{3.10}$$

where $k(0)$, $A(0)$ are free constants, determining the amplitude and wavelength at the initial point of time. The sign of q in equation (3.9) determines the stability of the solution $\psi = 0$. When $q > 0$, the solution is stable, the amplitude A is decreasing; otherwise, the solution is unstable, the amplitude A is increasing. However, for

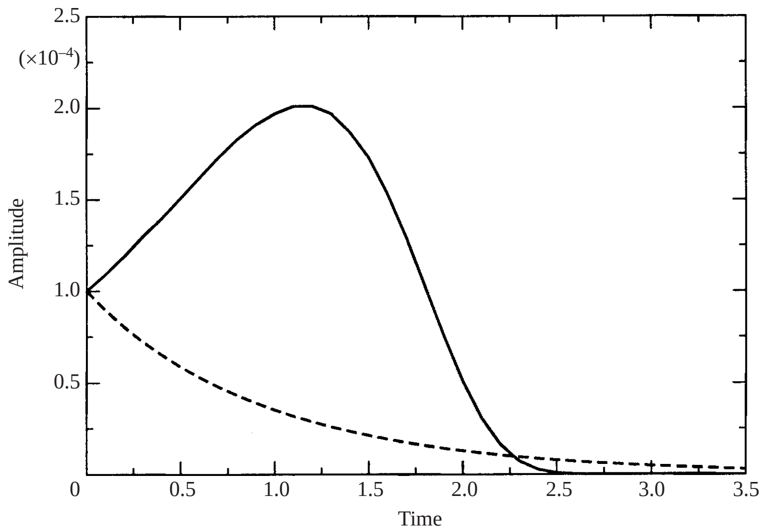

FIGURE 2. Amplitude for $q < 0$ (solid) and $q > 0$ (dashed).

$q < 0$ the solution is unstable only until $t = (1/(-2q)) \ln |q/(\nu k^2(0))|$, after which the amplitude decreases rapidly, owing to dissipation (figure 2).

4. Results and discussion

4.1. *Stability analysis*

Let us consider the case when the flow function perturbation has the following form:

$$\delta\psi = \sum_{i=1}^{N} \frac{A_i(t)}{k_i^2(t)} \cos(k_{i1}(t)x + k_{i2}(t)y), \tag{4.1}$$

provided that $k_{i1}^2 + k_{i2}^2 \equiv k^2 \ \forall i$.

LEMMA 1. *If $k_{i1}^2 + k_{i2}^2 \equiv k^2 \ \forall i$ then equation* (4.1) *is the exact solution of equation* (3.3), *with $k(t)$, $A_i(t)$ defined from expression* (3.9).

Proof. The proof is based on the reduction of (3.3) to a linear equation (where the principle of superposition is valid). The property of the Poisson brackets is $\{\psi, \psi\} \equiv 0$. We find the vorticity (since $k_{i1}^2 + k_{i2}^2 \equiv k^2 \ \forall i$, it is possible to carry out the summation in (4.1)) from:

$$\omega \equiv \Delta\psi = -\sum_{i=1}^{N} A_i \cos(k_{i1}(t)x + k_{i2}(t)y) \equiv -k^2(t)\psi; \tag{4.2}$$

$k^2(t)$ does not depend on the spatial coordinate. Therefore

$$\{\psi, \omega\} = \{\psi, -k^2(t)\psi\} = -k^2(t)\{\psi, \psi\} = 0, \tag{4.3}$$

which proves the lemma. □

REMARK 1. *If $q > 0$, the solution is stable, with both the amplitude and the wave-number k decreasing in the course of time. Otherwise if $q < 0$, the solution is unstable.*

However, the amplitude increases until $t = \sum_{i=1}^{N}(1/(-2q))\ln|q/(\nu k^2(0))|$, *after which owing to dissipation it decreases rapidly. The wavenumber k increases in the course of time. The new and interesting fact which has been discovered in the course of this research is that the wavenumber k, corresponding to the time* $t = \sum_{i=1}^{N}(1/(-2q))\ln|q/(\nu k^2(0))|$, *is not dependent on the initial conditions and is equal to* $k = \sqrt{-q/\nu}$.

It should be noted that in each of the cases investigated $q > 0$ corresponds to the situation when the disks are moving towards each other and $q < 0$ to the situation when the disks are moving apart.

Remark 2. *Note that if in equation* (4.1) $N = 1$, *then the results obtained by Craik* (1989) *are retrieved. This case corresponds to a perturbation in a form of one plane wave. The case when* $N > 1$ *corresponds to plane-wave superposition, which can* (*for special conditions for wavenumber and amplitude* (*Chandrasekhar* 1997)) *reduce to the appearance of different space structures.*

4.2. *Stationary solutions in the form of jets*

So far, the solution $\psi = 0$, corresponding to the liquid motion near a stagnation point, has been considered. It is also relevant to find and examine other stationary solutions, such as jets. We consider the flow function in the following form:

$$\psi = xF(y) + \phi(y). \tag{4.4}$$

In this case (3.3) takes the following form:

$$\phi_x\omega_y - \phi_y\omega_x = -q(2\omega + y\omega_y + x\omega_x) + \nu\Delta\omega, \tag{4.5}$$

and since

$$\omega = xF''(y) + \phi''(y), \tag{4.6}$$

equation (4.5) can be rewritten as

$$F(xF''' + \phi''') - (xF' - \phi')F'' = -q(2(xF'' + \phi'') + y(xF''' + \phi''') + xF'') + \nu(xF^{\mathrm{iv}} + \phi^{\mathrm{iv}}), \tag{4.7}$$

where a prime denotes a derivative with respect to the argument (here y) and a superscript iv denotes the fourth derivative. Equating groups of terms with the same x-powers it is possible to obtain the following system:

$$FF''' - F'F'' = -q(2F'' + yF''' + F'') + \nu F^{\mathrm{iv}}, \tag{4.8}$$

$$F\phi''' + \phi'F'' = -q(2\phi'' + y\phi''') + \nu\phi^{\mathrm{iv}}. \tag{4.9}$$

We consider the particular case when the analytical solution of equation (4.8) is $F = ay$. In this case (4.9) will take the following form:

$$ay\phi''' + q(2\phi'' + y\phi''') = \nu\phi^{\mathrm{iv}}. \tag{4.10}$$

After some mathematical transformations and integrating twice, we obtain the following equation:

$$\nu\phi'' = (a + q)y\phi' - 2a\phi, \tag{4.11}$$

which has the form of Hermite's differential equation when two conditions are satisfied: $(a + q)/\nu = 2$ and a/ν is a non-negative integer. The solutions of this equation have the following form:

$$\phi = \frac{\mathrm{d}^n}{\mathrm{d}y^n}\left(A\exp\left(\frac{q}{(3+n)\nu}y^2\right)\right), \tag{4.12}$$

where the relation between a and q is

$$a = -\frac{1+n}{3+n}q, \quad n \in [0, \infty]. \tag{4.13}$$

Thus the solutions of equation (3.3) can be written as

$$\psi = -\frac{1+n}{3+n}qxy + \frac{\mathrm{d}^n}{\mathrm{d}y^n}\left(A \exp\left(\frac{q}{(3+n)\nu}y^2\right)\right). \tag{4.14}$$

In equation (4.14), the first term denotes the liquid motion corresponding to the potential flow component, and the second term represents the jet behaviour (non-potential flow component). Since $q < 0$, $n > 0$, $\nu > 0$, it can be seen that this second term approaches zero for $y \to \pm\infty$.

4.3. *Transient solution in the form of logarithmic spiral jets*

As well as the above stationary solutions for jets, it is also possible to obtain solutions in the form of rotating spiral jets. In the context of the present paper let us rewrite (3.3) in the cylindrical coordinate system:

$$\left(\frac{\partial}{\partial t} - qr\frac{\partial}{\partial r}\right)(r^2\omega) + \frac{\partial \psi}{\partial \varphi}r\frac{\partial \omega}{\partial r} - \frac{\partial \omega}{\partial \varphi}r\frac{\partial \psi}{\partial r} = \nu\left(\frac{\partial^2 \omega}{\partial \varphi^2} + r\frac{\partial}{\partial r}r\frac{\partial \omega}{\partial r}\right), \tag{4.15}$$

where

$$\omega = \Delta\psi = \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}. \tag{4.16}$$

Then

$$\psi = -4\nu\varphi + F(qt + \ln r + n\varphi). \tag{4.17}$$

The expression for the function F is

$$-4\nu F' + nF'F' + \nu(1+n^2)F''' = \text{const}, \tag{4.18}$$

where the derivation takes an argument $\xi = qt + \ln r + n\varphi$.

If we introduce the variable $U = F'$, we obtain

$$U'' - \frac{4}{1+n^2}U + \frac{n}{\nu(1+n^2)}U^2 = \text{const}, \tag{4.19}$$

$$\frac{(U')^2}{2} - \frac{2U^2}{1+n^2} + \frac{nU^3}{3\nu(1+n^2)} = c_1U + c_2. \tag{4.20}$$

It should be noted that periodic solutions for the function U should exist, and the period for ξ should be equal to $\xi_N = \xi + 2\pi Nn$, where $N \in [0, \infty]$. Now it is possible to write

$$\frac{(U')^2}{2} = c_2 + c_1U + \frac{2}{1+n^2}U^2 - \frac{n}{3\nu(1+n^2)}U^3. \tag{4.21}$$

The expressions for the velocity components are

$$v_r = -qr + \frac{4\nu}{r} + \frac{n}{r}U(qt + \ln r + n\varphi), \tag{4.22}$$

$$v_\varphi = -\frac{1}{r}U(qt + \ln r + n\varphi), \tag{4.23}$$

$$v_z = 2qz. \tag{4.24}$$

The three-dimensional non-stationary solution for logarithmic spiral jets is obtained, albeit as a function of the newly introduced variable $U = F'$, which has to be determined numerically. Note that for certain values of v and n a solution of equation (4.21) can be found as an elliptic integral of the second type.

The case when $q = 0$ was considered by Netreba (1988).

Finally, we found as a result of the investigation that the structures rotate proportionally to the relative velocity of the disks.

5. Conclusions

In this paper, we have considered viscous flow between two parallel disks. These disks can move towards each other or in opposite directions. The liquid flow is described by the Navier–Stokes equations, for which some analytical solutions have been obtained. Furthermore, an instability analysis has been performed by assuming a perturbation in the form of a superposition of plane waves.

When the disks are moving towards each other, the liquid flow has a laminar behaviour. On the other hand, when the disks are moving apart from each other the results were as follows.

(*a*) The solution of the Navier–Stokes equations near the stagnation point is obtained. For this specific case, an instability is found, but this instability disappears after a certain time interval. It has been proved that with the evolution of perturbation of a certain type, wave structures with different configurations are formed.

(*b*) Stationary solutions in the form of jets were found.

(*c*) Transient solutions in the form of logarithmic spiral jets were found.

This paper was supported by united program CNRS-RFBR (grant PICS 1170/RFBI 01-0122005 CNTI), and partially supported by US Civil Research & Development Foundation for the independent state former Soviet Union (CRDF) through grant No. PE-009-0.

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